Induktion Tutorium

1) $\sum_{k=1}^{n-1} \frac{1}{k} \geq \ln(n)$ $\quad n \geq 2$ $\qquad$ Hinweis: $\ln(x) \leq x-1$

(I) links ($n=2$): $\sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} = 1$ $\qquad$ rechts: $\ln(2)$

$1 \geq \ln(2)$ o.k.

(II) Es gelte für ein $n \in \mathbb{N}$, $n \geq 2$

$$\sum_{k=1}^{n-1} \frac{1}{k} \geq \ln(n)$$

(III) $n \to n+1$

z.z. $\sum_{k=1}^{n} \frac{1}{k} \geq \ln(n+1)$

links: $\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n} \overset{\text{II}}{\geq} \ln(n) + \frac{1}{n} \overset{?}{\geq} \ln(n+1)$

kläre?

$$\ln(n) + \frac{1}{n} \geq \ln(n+1) \quad | -\ln(n)$$

$$\Leftrightarrow \frac{1}{n} \geq \ln(n+1) - \ln(n)$$

$$\Leftrightarrow \frac{1}{n} \geq \ln\left(\frac{n+1}{n}\right)$$

$$\Leftrightarrow \frac{1}{n} + 1 - 1 \geq \ln\left(1 + \frac{1}{n}\right)$$

$$\Leftrightarrow 1 + \frac{1}{n} - 1 \geq \ln\left(1 + \frac{1}{n}\right)$$

$$\Leftrightarrow \ln\underbrace{\left(1 + \frac{1}{n}\right)}_{x} \leq \underbrace{1 + \frac{1}{n}}_{x} - 1$$

$\Rightarrow$ o.k. Hinweis

Induktion Tutorium

1) $n! \geq 3\left(\frac{n}{3}\right)^n$ Hinweis: $\left(1+\frac{1}{n}\right)^n \leq 3$

(I) $n=1$

links: $1! = 1$ rechts: $3\cdot\left(\frac{1}{3}\right)^1 = 1$

$\Rightarrow 1 \geq 1$ o.k

(II) Es gelte für ein $n \in \mathbb{N}$

$$n! \geq 3\left(\frac{n}{3}\right)^n$$

(III) z.z. $n \rightarrow n+1$

$$(n+1)! \geq 3\left(\frac{n+1}{3}\right)^{n+1}$$

links: $(n+1)! = n!(n+1) \underset{\text{(II)}}{\geq} 3\left(\frac{n}{3}\right)^n(n+1) \underset{?}{\geq} 3\left(\frac{n+1}{3}\right)^{n+1}$

kläre? $\quad 3\left(\frac{n}{3}\right)^n(n+1) \geq 3\left(\frac{n+1}{3}\right)^{n+1} \quad |:3$

$$\Leftrightarrow \left(\frac{n}{3}\right)^n(n+1) \geq \left(\frac{n+1}{3}\right)^{n+1}$$

$$\Leftrightarrow \frac{n^n}{3^n}(n+1) \geq \frac{(n+1)^{n+1}}{3^{n+1}}$$

$$\Leftrightarrow \frac{n^n}{3^n}(n+1) \geq \frac{(n+1)^n\cdot(n+1)}{3^n\cdot 3} \quad |\cdot 3^n$$

$$\Leftrightarrow n^n(n+1) \geq \frac{(n+1)^n\cdot(n+1)}{3} \quad |\cdot 3$$

$$\Leftrightarrow 3\cdot n^n(n+1) \geq (n+1)^n\cdot(n+1) \quad |:(n+1)$$

$$\Leftrightarrow 3\cdot n^n \geq (n+1)^n \quad |:n^n$$

$$\Leftrightarrow 3 \geq \left(\frac{n+1}{n}\right)^n$$

$$\Leftrightarrow \left(\frac{n}{n}+\frac{1}{n}\right)^n \leq 3$$

$$\Leftrightarrow \left(1+\frac{1}{n}\right)^n \leq 3$$ o.k, hinweise

Vollständiges Video:

https://www.youtube.com/watch?v=LvRQa-wStY8

Induktion Tutorium

1) $\sum_{k=2}^{n} (k-1) \ln\left(\frac{k}{k-1}\right) = \ln\left(\frac{n^n}{n!}\right), \quad n \geq 2$

links: $\sum_{k=2}^{2} (k-1) \ln\left(\frac{k}{k-1}\right) = (2-1) \ln\left(\frac{2}{1}\right)$
$= \ln(2)$

rechts: $\ln\left(\frac{2^2}{2!}\right) = \ln\left(\frac{4}{2}\right) = \ln(2)$
$\Rightarrow \ln(2) = \ln(2)$

(II) Es gelte für ein $n \in \mathbb{N}, n \geq 2$

$$\sum_{k=2}^{n} (k-1) \ln\left(\frac{k}{k-1}\right) = \ln\left(\frac{n^n}{n!}\right)$$

(III) z.z. $\sum_{k=2}^{n+1} (k-1) \ln\left(\frac{k}{k-1}\right) = \ln\left(\frac{(n+1)^{n+1}}{(n+1)!}\right)$

umformung

$\ln\left(\frac{(n+1)^{n+1}}{(n+1)!}\right)$

$\ln\left(\frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{(n+1)}\, n!}\right)$

$\ln\left(\frac{(n+1)^n}{n!}\right)$

links: $\sum_{k=1}^{n+1} (k-1) \ln\left(\frac{k}{k-1}\right) = \sum_{k=2}^{n} (k-1) \ln\left(\frac{k}{k-1}\right) + n \cdot \ln\left(\frac{n+1}{n}\right)$

$\overset{\text{II}}{=} \ln\left(\frac{n^n}{n!}\right) + n \cdot \ln\left(\frac{n+1}{n}\right)$

$= \ln\left(\frac{n^n}{n!}\right) + \ln\left(\frac{n+1}{n}\right)^n$

$= \ln\left(\frac{\cancel{n^n}}{n!} \cdot \frac{(n+1)^n}{\cancel{n^n}}\right)$

$= \ln\left(\frac{(n+1)^n}{n!}\right) \Rightarrow$ o.k

Induktion Tutorium

1) $\sum_{k=1}^{2n} \left(\frac{-1}{k}\right)^{k+1} = \sum_{k=n+1}^{2n} \frac{1}{k}$

links: $\sum_{k=1}^{2} \left(\frac{-1}{k}\right)^{k+1} = \left(\frac{-1}{1}\right)^{1+1} + \left(\frac{-1}{2}\right)^{2+1} = 1 - \frac{1}{2} = \frac{1}{2}$

rechts: $\sum_{k=2}^{2} \frac{1}{k} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$ o.k

(II) Es gelte für ein $n \in \mathbb{N}$,

$$\sum_{k=1}^{2n} \left(\frac{-1}{k}\right)^{k+1} = \sum_{k=n+1}^{2n} \frac{1}{k}$$

(III) $n \to n+1$

z.z. $\sum_{k=1}^{2n+2} \left(\frac{-1}{k}\right)^{k+1} = \sum_{k=n+2}^{2n+2} \frac{1}{k}$

links: $\sum_{k=1}^{2n+2} \left(\frac{-1}{k}\right)^{k+1} = \sum_{k=1}^{2n} \left(\frac{-1}{k}\right)^{k+1} + \underbrace{\left(\frac{-1}{2n+1}\right)^{\overset{2n+1+1}{2n+2}}}_{+} + \underbrace{\left(\frac{-1}{2n+2}\right)^{\overset{2n+2+1}{2n+3}}}_{-}$

$\overset{IV}{=} \sum_{k=n+1}^{2n} \frac{1}{k} + \left(\frac{1}{2n+1}\right) - \left(\frac{1}{2n+2}\right)$

$= \sum_{k=n+2}^{2n} \frac{1}{k} + \left(\frac{1}{2n+1}\right) - \left(\frac{1}{2n+2}\right) + \left(\frac{1}{n+1}\right)$

$= \sum_{k=n+2}^{2n} \frac{1}{k} + \left(\frac{1}{2n+1}\right) - \frac{1(n+1)}{(2n+2)(n+1)} + \frac{1(2n+2)}{(n+1)(2n+2)}$

$= \sum_{k=n+2}^{2n} \frac{1}{k} + \left(\frac{1}{2n+1}\right) \frac{-n-1+2n+2}{(2n+2)(n+1)}$

$= \sum_{k=n+2}^{2n} \frac{1}{k} + \left(\frac{1}{2n+1}\right) + \frac{\cancel{(n+1)}}{\cancel{(n+1)}(2n+2)} = \sum_{k=n+2}^{2n} \frac{1}{k} + \frac{1}{2n+1} + \frac{1}{2n+2}$

$= \sum_{k=n+2}^{2n+2} \frac{1}{k}$ ok

Vollständiges Video:

https://www.youtube.com/watch?v=AR9H-eerkUY

Neue Version mit Korrekturen

$$\sum_{k=1}^{2n} \left(\frac{-1}{k}\right)^{k+1} \geq \sum_{k=n+1}^{2n} \frac{1}{k}$$

<u>links</u>: $n=1$

$$\sum_{k=1}^{2} \left(\frac{-1}{1}\right)^{2} + \left(\frac{-1}{2}\right)^{3} = 1 + \frac{-1}{8}$$

$$= 1 - \frac{1}{8}$$

$$= \frac{8}{8} - \frac{1}{8} = \frac{7}{8}$$

<u>rechts</u>: $n=1$

$$\sum_{k=n+1}^{2} \frac{1}{k} = \sum_{k=2}^{2} \frac{1}{2} \qquad \frac{7}{8} \geq \frac{1}{2}$$

(II) Es gelte für ein $n \in \mathbb{N}$,

$$\sum_{k=1}^{2n} \left(\frac{-1}{k}\right)^{k+1} \geq \sum_{k=n+1}^{2n} \frac{1}{k}$$

(III) $n \to n+1$

z.z. $$\sum_{k=1}^{2n+2} \left(\frac{-1}{k}\right)^{k+1} \geq \sum_{k=n+2}^{2n+2} \frac{1}{k}$$

<u>links</u>: $$\sum_{k=1}^{2n+2} \left(\frac{-1}{k}\right)^{k+1} = \sum_{k=1}^{2n} \left(\frac{-1}{k}\right)^{k+1} + \left(\frac{-1}{2n+1}\right)^{2n+2} + \left(\frac{-1}{2n+2}\right)^{2n+3}$$

$$\overset{(II)}{=} \sum_{k=n+1}^{2n} \left(\frac{1}{k}\right) + \left(\frac{1}{2n+1}\right) - \left(\frac{1}{2n+2}\right) = \sum_{k=n+2}^{2n} \left(\frac{1}{k}\right) + \left(\frac{1}{2n+1}\right) - \left(\frac{1}{2n+2}\right) + \left(\frac{1}{n+1}\right)$$

$$= \sum_{k=n+2}^{2n} \left(\frac{1}{k}\right) + \left(\frac{1}{2n+1}\right) - \frac{(n+1)}{(2n+2)(n+1)} + \frac{(2n+2)}{(n+1)(2n+2)}$$

$$= \sum_{k=n+2}^{2n} \left(\frac{1}{k}\right) + \left(\frac{1}{2n+1}\right) - \frac{n-1+2n+2}{(2n+2)(n+1)} = \sum_{k=n+2}^{2n} \left(\frac{1}{k}\right) + \left(\frac{1}{2n+1}\right) + \frac{\cancel{(n+1)}}{(2n+2)\cancel{(n+1)}}$$

$$= \sum_{k=n+2}^{2n} \left(\frac{1}{k}\right) + \left(\frac{1}{2n+1}\right) + \frac{1}{2n+2} = \sum^{2n+2} \frac{1}{k}$$

Induktion Tutorium

1) $x^n - nx + n - 1 \geq 0 \quad \forall x > 0$ und $n \in \mathbb{N},\ n \geq 2$

I) $n = 2$

$$x^2 - 2x + 1 \geq 0$$

$$\Leftrightarrow x^2 - 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 1 \geq 0$$

$$\Leftrightarrow (x-1)^2 \geq 0 \quad \checkmark \text{ o.k.}$$

II) Es gelte für ein $n \in \mathbb{N},\ n \geq 2$

$$x^n - nx + n - 1 \geq 0$$

III) $n \to n+1$

z.z. $x^{n+1} - (n+1)x + n + 1 - 1 \geq 0$

links: $x^{n+1} + x^n - x^n - (n+1)x + n + 1 - 1 \geq 0$

$$= x^{n+1} + x^n - x^n - nx - x + n + 1 - 1 \geq 0$$

$$= x^n - nx + n - 1 + x^{n+1} - x^n - x + 1 \geq 0 \quad \downarrow \text{ Sortieren}$$

$$\overset{II}{=} 0 + x^{n+1} - x^n - x + 1 \overset{?}{\geq} 0$$

kläre ? $x^n \cdot x - x^n - x + 1 \geq 0 \quad \downarrow$ x^n ausklammern, ~~vor die~~ $-$ ausklammern

$$\Leftrightarrow x^n(x-1) - (x-1) \geq 0 \quad \downarrow \text{ } x-1 \text{ ausklammern}$$

$$\Leftrightarrow (x-1)(x^n - 1) \overset{?}{\geq} 0$$

Fall 1: $\oplus \cdot \oplus = +$

$$(x-1) \geq 0$$

$$\Leftrightarrow x \geq 1$$

$$\Rightarrow x^n > 1$$

$$\Rightarrow x^n - 1 \geq 0 \quad \checkmark$$

also stimmt
beide Positiv

Fall 2: $\ominus \cdot \ominus$

$$(x-1) < 0$$

$$\Leftrightarrow x < 1$$

$$\Rightarrow x^n < 1$$

$$\Rightarrow x^n - 1 < 0 \quad \checkmark$$

stimmt, beide negativ

Induktion Tutorium

1) $\sum_{k=1}^{n-1} \frac{1}{k} \geq \ln(n)$, $n \in \mathbb{N}, n \geq 2$

Hinweis: $\ln x \leq x - 1$

(I) $\sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1}$ $\qquad \ln(n) = \ln(2)$

$1 \geq \ln(2)$

(II) Es gelte für ein $n \in \mathbb{N}, n \geq 2$

$$\sum_{k=1}^{n-1} \frac{1}{k} \geq \ln(n)$$

(III) z.z. $n \to n+1$

$$\sum_{k=1}^{n} \frac{1}{k} \geq \ln(n+1)$$

links: $\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n} \overset{\text{II}}{\geq} \ln(n) + \frac{1}{n} \overset{?}{\geq} \ln(n+1)$

kläre?:

$$\ln(n) + \frac{1}{n} \geq \ln(n+1)$$

$$\Leftrightarrow \quad \frac{1}{n} \geq \ln(n+1) - \ln(n)$$

$$\Leftrightarrow \quad \frac{1}{n} \geq \ln\left(\frac{n+1}{n}\right)$$

$$\cancel{\frac{1}{n} \geq \ln\left(\frac{n+1}{n+1}\right)}$$

$$\Leftrightarrow \quad \frac{1}{n} \geq \ln\left(\frac{n}{n} + \frac{1}{n}\right)$$

$$\Leftrightarrow \quad \ln\left(1 + \frac{1}{n}\right) \overset{?}{\leq} \frac{1}{n}$$

$\Rightarrow$ o.k., Hinweis

Ungleichungen mit Induktion und Zahlenfolge

Vollständiges Video:

https://www.youtube.com/watch?v=5fI3xD2rRkg

a) Weisen Sie nach:

$$(1+x)^n \geq 1+nx \qquad x \geq -1,\ n \in \mathbb{N}$$

I $n=1$: $1+x \geq 1+1x$

$1+x \geq 1+x$ ✓

II Es gelte für ein $N \in \mathbb{N}$, $x \geq -1$

$$(1+x)^n \geq 1+nx$$

III z.z. $(1+x)^{n+1} \geq 1+(n+1)x$

$$(1+x)^{n+1} = (1+x)^n \cdot 1+x \underset{\text{II}}{\geq} 1+nx \cdot 1+x \underset{?}{\geq} 1+(n+1)x$$

kläre ?

$$1+nx \cdot 1+x \geq 1+(n+1)x$$

$$\Leftrightarrow (1+nx) \cdot (1+x) \geq 1+nx+x$$

$$\Leftrightarrow 1+x+nx+nx^2 \geq 1+nx+x \qquad |-nx$$

$$\Leftrightarrow 1+x+nx^2 \geq 1+x \qquad |-1$$

$$\Leftrightarrow x+nx^2 \geq x$$

$$\Leftrightarrow nx^2 \geq 0 \qquad \text{wahr}$$

a) $(1+x)^n \geq 1+nx$

b) Schließen Sie aus a) auf

$$\left(1-\frac{1}{n^2}\right)^n \geq 1-\frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$\boxed{x = \frac{-1}{n^2}}$$

- $\left(1-\frac{1}{n^2}\right)^n \geq 1+n\cdot\left(\frac{-1}{n^2}\right)$

$\Leftrightarrow \left(1-\frac{1}{n^2}\right)^n \geq 1-\frac{\cancel{n}}{n^{\cancel{2}}}$

$\left(1-\frac{1}{n^2}\right)^n \geq \left(1-\frac{1}{n}\right)$ q.e. bewiesen

c) Zeigen Sie die Folge : $a_n = \left(1+\frac{1}{n}\right)^n$ ist Monoton ↑

Bilde $\frac{a_n}{a_{n-1}}$ und vervollständigen Sie die Rechnung

$$\frac{a_n}{a_{n-1}} = \ldots = \left(\frac{\left(1+\frac{1}{n}\right)}{1+\frac{1}{n-1}}\right)^n \left(1+\frac{1}{n-1}\right) = \left(\frac{n^2-1}{n^2}\right)^n \frac{n}{n-1} \cdots \geq 1$$

indem Sie an geeigneter Stelle b) nutzen

$$\frac{a_n}{a_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^n} \cdot \left(1+\frac{1}{n-1}\right) ✓$$

$$= \left(\frac{\left(\frac{n}{n}+\frac{1}{n}\right)^n}{\left(\frac{n-1}{n-1}+\frac{1}{n-1}\right)^n}\right) \cdot \left(\frac{n-1}{n-1}+\frac{-1}{n-1}\right) = \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^n} \cdot \left(\frac{n}{n-1}\right)$$

$$= \frac{(n+1)\cdot(n-1)}{n^2} \cdot \left(\frac{n}{n-1}\right) = \left(\frac{n^2 \cancel{-n} \cancel{+n} - 1}{n^2}\right)^n \cdot \frac{n}{n-1} = \left(\frac{n^2-1}{n^2}\right)^n \cdot \frac{n}{n-1} ✓$$

$$= \left(1-\frac{1}{n^2}\right)^n \cdot \frac{n}{n-1} \overset{\text{einsetzen}}{=} \left(1-\frac{1}{n}\right)^n \cdot \frac{n}{n-1} = \left(\frac{n}{n}-\frac{1}{n}\right)^n \cdot \frac{n}{n-1}$$

$$= \frac{n-1}{\cancel{n}} \cdot \frac{\cancel{n}}{n-1} = 1 \geq 1 ✓$$

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Ungleichungen mit Induktion und Zahlenfolge

Vollständiges Video:

https://www.youtube.com/watch?v=4xic_3Zja98

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n} \quad n \in \mathbb{N}$$

rechts: $2 - \frac{1}{1} = 2 - 1 = 1$ o.k.

(I) $n = 1$ $\quad \sum_{k=1}^{1} \frac{1}{k^2} = \frac{1}{1^2} = 1$

(II) Es gelte für ein $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

(III) $n \to n+1$

z.z. $\quad \sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}$

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \underset{\text{II}}{\leq} 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \underset{?}{\leq} 2 - \frac{1}{n+1}$$

kläre? $\quad 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1} \quad | -2$

$$\Leftrightarrow \quad -\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1} \quad \left| + \frac{1}{n}\right.$$

$$\Leftrightarrow \quad \frac{1}{(n+1)^2} \leq -\frac{1}{n+1} + \frac{1}{n}$$

$$\Leftrightarrow \quad \frac{1}{(n+1)^2} \leq \frac{-n}{(n+1)n} + \frac{n+1}{(n+1)n}$$

$$\Leftrightarrow \quad \frac{1}{(n+1)^2} \leq \frac{\cancel{-n} + \cancel{n} + 1}{(n+1)n}$$

$$\Leftrightarrow \quad \frac{1}{(n+1)^2} \leq \frac{1}{(n+1)n}$$

$$\Leftrightarrow \quad \frac{1}{n^2+2n+1} \leq \frac{1}{n^2+n} \quad \text{wahr}$$

o.k.

n^2+2n+1 ist größer als $\frac{1}{n^2+n}$

deshalb gesamt kleiner

b) sei $s_n := \sum_{k=1}^{n} \frac{1}{k^2} \quad n \in \mathbb{N}$

zeigen Sie, dass $\lim_{n \to \infty} s_n = s$

existiert und die Relation

$1 \leq s \leq 2$ erfüllt.

$\Rightarrow \quad s_n \leq 2 - \frac{1}{n} \leq 2$

$s_n \geq 1$? ja!

$1 \leq s_n \leq 2$

$\Rightarrow \quad 1 \leq s_n \leq 2$

$1 \leq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 \quad$ q.e.d.

Induktion Tutorium

$$\prod_{j=1}^{n} (1+x_j) \geq 1+\sum_{j=1}^{n} x_j \qquad \forall x \geq 0$$

linkes: $\prod_{j=1}^{1} (1+x_j) = 1+x_j$ rechts: $1+\sum_{j=1}^{1} x_j$

$\Rightarrow 1+x_j \geq 1+x_j$ o.k. $= 1+x_j$

(II) Es gelte für ein $n \in \mathbb{N}$, $\prod_{j=1}^{n} (1+x_j) \geq 1+\sum_{j=1}^{n} x_j$

(III) $n \to n+1$

z.z. $\prod_{j=1}^{n+1} (1+x_j) \geq 1+\sum_{j=1}^{n+1} x_j$

linkes: $\prod_{j=1}^{n+1} (1+x_j) = \prod_{j=1}^{n} (1+x_j)\cdot(1+x_{n+1}) \overset{\text{II}}{=} 1+\sum_{j=1}^{n} x_j \cdot (1+x_{n+1})$

$$1+\sum_{j=1}^{n} x_j \cdot (1+x_{n+1}) \underset{?}{\geq} 1+\sum_{j=1}^{n+1} x_j$$

kläre ? $\left(1+\sum_{j=1}^{n} x_j\right) \cdot (1+x_{n+1}) \geq 1+\sum_{j=1}^{n+1} x_j$

$$\Leftrightarrow 1+x_{n+1}+\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} x_j \cdot x_{n+1} \geq 1+\sum_{j=1}^{n+1} x_j$$

$$\Leftrightarrow 1+\sum_{j=1}^{n+1} x_j + \underbrace{\sum_{j=1}^{n} x_j \cdot x_{n+1}}_{\geq 0} \geq 1+\sum_{j=1}^{n+1} x_j$$

$\rightarrow$ o.k., weil $\forall x \geq 0$

Grenzwerte Tutorium

$$\lim_{n\to\infty} \frac{x^{n^2}}{(n^{2n})} \quad x>0$$

$$\frac{x^{n^2}}{(n^{2n})}$$

$$= e^{\ln\left(\frac{x^{n^2}}{n^{2n}}\right)}$$

$$= e^{\ln(x^{n^2}) - \ln(n^{2n})}$$

$$= e^{n^2 \ln(x) - 2n\ln(n)}$$

$$= e^{n^2\left(\ln(x) - \frac{2\cancel{n}\ln(n)}{n^{\cancel{2}}}\right)}$$

$$= e^{n^2\left(\ln(x) - 2\cdot\underbrace{\frac{\ln(n)}{n}}_{0}\right)}$$

$$\xrightarrow[n\to\infty]{} \begin{cases} e^{\infty} = \infty & , x>1 \\ e^{-\infty} = 0 & , 1>x>0 \\ \frac{1}{\infty} = 0 & , x=1 \end{cases}$$

$$\cancel{\lim} \quad \lim_{n\to\infty} \left(\frac{x^{n^2}}{n^{2n}}\right) = \frac{1}{\infty} = 0$$

Formelsammlung

Grenzwerte Tutorium

$$\lim_{n \to \infty} \frac{n^n}{(2n)!}$$

$$\frac{n^n}{(2n)!}$$

$$= \frac{\overbrace{n \cdot n \cdot n \cdot n \cdot n \cdot n \cdot n}^{n\text{-mal}}}{\underbrace{1 \; 2 \; 3 \ldots (n-2)(n-1)\; n}_{:n} \; (n+1)(n+2)\ldots(1n+1)(1n+2)\ldots(2n-2)(2n-1)(2n)}$$

$$= \frac{1}{(n+1)(n+2)\ldots(1n+1)(1n+2)\cdots(2n-2)(2n-1)(2n)}$$

$$\xrightarrow[n \to \infty]{} \frac{1}{\infty} = 0$$

Grenzwerte Tutorium

$$\lim_{n\to\infty} \frac{a^n - a^{-n}}{a^n + a^{-n}} \qquad a \neq 1$$

$$\frac{a^n - a^{-n}}{a^n + a^{-n}} = \frac{\frac{a^n}{a^{-n}} - \frac{a^{-n}}{a^{-n}}}{\frac{a^n}{a^{-n}} + \frac{a^{-n}}{a^{-n}}}$$

$$= \frac{a^{2n} - 1}{a^{2n} + 1}$$

1 Fall: $|a| < 1$

$$\frac{a^{2n} - 1}{a^{2n} + 1} \quad \lim_{n\to\infty} \frac{0 - 1}{0 + 1} = \frac{1}{1} = 1$$

2 Fall: $a > 1$

$$\frac{a^{2n} - 1}{a^{2n} + 1} = \frac{\frac{a^{2n}}{a^{2n}} - \frac{1}{a^{2n}}}{\frac{a^{2n}}{a^{2n}} + \frac{1}{a^{2n}}} = \frac{1 - \frac{1}{a^{2n}}}{1 + \frac{1}{a^{2n}}}$$

$$\lim_{n\to\infty} \frac{1 - 0}{1 + 0} = \frac{1}{1} = 1$$

3 Fall: $a = 1$

$$\cancel{\frac{a^n - a^{-n}}{a^n + a^{-n}} \quad \lim_{n\to\infty} \quad 0} \quad ?$$

$a \neq 1$

Grenzwerte Tutorium

$$\lim_{n\to\infty} \frac{\ln^2((2n)^{2n})}{n!}$$

$$\frac{\ln^2((2n)^{2n})}{n!}$$

$$= \frac{\ln(2n)^{2n} \cdot \ln(2n)^{2n}}{n!}$$

$$= \frac{2n\ln(2n) \cdot 2n\ln(2n)}{n!}$$

$$= \frac{2n\cdot 2n\,\frac{\ln(2n)}{2n} \cdot 2n\cdot 2n\,\frac{\ln(2n)}{2n}}{n!}$$

$$\cancel{= 16n^4 \cdot \frac{\ln(2n)}{2n} \cdot \frac{\ln(2n)}{2n}}$$

$$\cancel{\xrightarrow[n\to\infty]{} 16n^4 \cdot 0 \cdot 0 = 0}$$

$$= \frac{16n^4}{n!} = \frac{16n^{3}}{n(n-1)!} = \frac{16n^3}{(n-1)!} = \frac{16}{(n-4)!}\cdot\frac{n}{(n-1)}\cdot\frac{n}{(n-2)}\cdot\frac{n}{n-3} \quad |:n^3$$

$$= \frac{16}{(n-4)!}\cdot\frac{\frac{n}{n}}{\left(\frac{n}{n}-\frac{1}{n}\right)}\cdot\frac{\frac{n}{n}}{\frac{n}{n}-\frac{2}{n}}\cdot\frac{\frac{n}{n}}{\frac{n}{n}-\frac{3}{n}} = \frac{16}{(n-4)!}\cdot\frac{1}{1-\frac{1}{n}}\cdot\frac{1}{1-\frac{2}{n}}\cdot\frac{1}{1-\frac{3}{n}}$$

$$\xrightarrow[n\to\infty]{} 0\cdot 1\cdot 1\cdot 1 = \underline{\underline{0}}$$

Grenzwerte Tutorium

$$(x^2-x+1)^n \Rightarrow x^2-x+1$$

$$x^2-x+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+1$$

$$\left(x-\tfrac{1}{2}\right)^2-\tfrac{1}{4}+\tfrac{4}{4}$$

$$\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}$$

Fall 1: $|x|<1$

$$\left|\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right|<1$$

Fall 1a:

$$\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}<1$$

$$\left(x-\tfrac{1}{2}\right)^2<\tfrac{1}{4}$$

Fall 1a.1:

$$\left(x-\tfrac{1}{2}\right)^{\cancel{2}}<\sqrt{\tfrac{1}{4}}$$

$$x<\sqrt{\tfrac{1}{4}}+\tfrac{1}{2}$$

$$x<\tfrac{1}{2}+\tfrac{1}{2}$$

$$\boxed{x<1} \rightarrow x\in(0,1)$$

Fall 1a.2:

$$-x+\tfrac{1}{2}<\sqrt{\tfrac{1}{4}}$$

$$-x<\tfrac{1}{2}-\tfrac{1}{2}$$

$$\boxed{x>0}$$

Fall 1b:

$$-\left(x-\tfrac{1}{2}\right)^2-\tfrac{3}{4}<1$$

$$-\left(x-\tfrac{1}{2}\right)^2<\tfrac{7}{4}$$

Fall 1b.1

$$-\left(x-\tfrac{1}{2}\right)<\tfrac{7}{4} \quad |\cdot(-1)$$

$$\left(x-\tfrac{1}{2}\right)>-\tfrac{7}{4} \quad \text{o.k}$$

stimmt.

2) $\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}=1$

$$\left(x-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$

Fall 1a:

$$\left(x-\tfrac{1}{2}\right)=\tfrac{1}{2}$$

$$\boxed{x=1}$$

$$x\in\{1,0\}$$

Fall 1b:

$$-x+\tfrac{1}{2}=\tfrac{1}{2}$$

$$-x=0$$

$$\boxed{x=0}$$

3) $\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}>1$

$$\left(x-\tfrac{1}{2}\right)^2>\tfrac{1}{4}$$

Fall 1a:

$$x-\tfrac{1}{2}>\tfrac{1}{2}$$

$$\boxed{x>1}$$

Fall 1b

$$-x+\tfrac{1}{2}>\tfrac{1}{2}$$

$$-x>0$$

$$\boxed{x<0}$$

$$x\in(-\infty,0)\cup(1,\infty)$$

4) $\left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4}\leq-1$

$$\left(x-\tfrac{1}{2}\right)^2\leq-\tfrac{1}{4}$$

$\lightning$

$$f(x)=\begin{cases} 0 & x\in(0,1) \\ 1 & x\in\{1,0\} \\ \infty & x\in(-\infty,0)\cup(1,\infty) \end{cases}$$

Grenzwerte Tutorium

$\cot = \left(\frac{\cos}{\sin}\right)$

$$\lim_{x \to 0} x^2 \cot(2x^2)$$

$$x^2 \cot(2x^2)$$

$$= x^2 \frac{\cos(2x^2)}{\sin(2x^2)}$$

$$= \frac{x^2}{\sin(2x^2)} \cdot \cos(2x^2)$$

$$= \frac{x^2 \cdot 2}{\sin(2x^2)} \cdot \frac{1}{2} \cdot \cos(2x^2)$$

$$= \underbrace{\frac{2x^2}{\sin(2x^2)}}_{1} \cdot \frac{1}{2} \cdot \underbrace{\cos(2x^2)}_{1} \xrightarrow[x \to 0]{} \frac{1}{2}$$

Grenzwerte Tutorium

$$\lim_{x \to \infty} \frac{\ln(2x)}{\ln(5x^2+1)}$$

$$\frac{\ln(2x)}{\ln(5x^2+1)}$$

$$= \frac{\ln(2x)}{\ln(x^2(5+\frac{1}{x^2})}$$

$$= \frac{\ln(2x)}{\ln(x^2) + \ln(5+\frac{1}{x^2})}$$

$$= \frac{\ln(2x)}{2\ln(x) + \ln(5+\frac{1}{x^2})}$$

$$= \frac{\frac{\ln(2)}{\ln(x)} + \frac{\ln(x)}{\ln(x)}}{2\frac{\ln(x)}{\ln(x)} + \frac{\ln(5+\frac{1}{x^2})}{\ln(x)}}$$

$$= \frac{\frac{\ln(2)}{\ln(x)} + 1}{2 \cdot 1 + \ln(\frac{5+\frac{1}{x^2}}{\ln(x)})} \xrightarrow[x \to \infty]{} \frac{0+1}{2+0} = \frac{1}{2}$$

Grenzwerte Tutorium

$$\lim_{x\to 0} \frac{\tan(2x)}{\tan(x)}$$

$$\frac{\tan(2x)}{\tan(x)}$$

$$= \frac{\frac{\sin(2x)}{\cos(2x)}}{\frac{\sin(x)}{\cos(x)}} = \frac{\sin(2x)\cdot\cos(x)}{\cos(2x)\cdot\sin(x)}$$

$$= \frac{\sin(2x)}{2x}\cdot 2\cancel{x}\cdot\frac{\cos(x)}{\cos(2x)}\cdot\frac{1}{\frac{\sin(x)}{x}\cdot\cancel{x}}$$

$$= \underbrace{\frac{\sin(2x)}{2x}}_{1}\cdot 2\cdot\underbrace{\frac{\cos(x)}{\cos(2x)}}_{1}\cdot\underbrace{\frac{1}{\frac{\sin x}{x}}}_{1}$$

$$\xrightarrow[x\to 0]{} 2$$

Grenzwerte Tutorium

$$\lim_{x \to 0^+} x \cdot \ln(e^x - 1)$$

$$x \cdot \ln(e^x - 1)$$

$$= \cancel{x \cdot \ln\left(\frac{(e^x-1)}{x}\right) \cdot x}$$

$$= x \cdot \ln\left(\frac{(e^x-1)}{x} \cdot x\right)$$

$$= x \cdot \left(\ln\left(\frac{(e^x-1)}{x}\right) + \ln(x)\right)$$

$$= \underset{0}{\underset{\approx}{x}} \cdot \underbrace{\ln\left(\underset{1}{\underbrace{\frac{(e^x-1)}{x}}}\right)}_{0} + \underbrace{x \cdot \ln(x)}_{0}$$

$$\xrightarrow[x \to 0^+]{} 0 \cdot 0 + 0 = \underline{\underline{0}}$$

Grenzwerte Tutorium

$$\lim_{n\to\infty} \frac{\sqrt{3n^4-n+7}-\sqrt{3n^4+7n^3-12}}{\sqrt{6n^4+80n^3}-\sqrt{6n^4-1}}$$

$$\frac{\sqrt{3n^4-n+7}-\sqrt{3n^4+7n^3-12}}{\sqrt{6n^4-80n^3}-\sqrt{6n^4-1}}$$

$$= \frac{\left(\sqrt{3n^4-n+7}-\sqrt{3n^4+7n^3-12}\right)\left(\sqrt{3n^4-n+7}+\sqrt{3n^4+7n^3-12}\right)\cdot\left(\sqrt{6n^4+80n^3}+\sqrt{6n^4-1}\right)}{\left(\sqrt{6n^4+80n^3}-\sqrt{6n^4-1}\right)\left(\sqrt{6n^4+80n^3}+\sqrt{6n^4-1}\right)\left(\sqrt{3n^4-n+7}+\sqrt{3n^4+7n^3-12}\right)}$$

$$= \frac{\left(\cancel{3n^4}-n+7-\cancel{3n^4}-7n^3+12\right)\left(\sqrt{6n^4+80n^3}+\sqrt{6n^4-1}\right)}{\left(6n^4+80n^3-6n^4+1\right)\left(\sqrt{3n^4-n+7}+\sqrt{3n^4+7n^3-12}\right)}$$

$$= \frac{\left(-n+19-7n^3\right)\left(\sqrt{6n^4+80n^3}+\sqrt{6n^4-1}\right)}{\left(80n^3+1\right)\left(\sqrt{3n^4-n+7}+\sqrt{3n^4+7n^3-12}\right)} \quad : n^3$$

$$= \frac{\left(\frac{-n}{n^3}+\frac{19}{n^3}-\frac{7n^3}{n^3}\right)\left(\sqrt{\frac{6n^4}{n^4}+\frac{80n^3}{n^4}}+\sqrt{\frac{6n^4}{n^4}-\frac{1}{n^4}}\right)}{\left(\frac{80n^3}{n^3}+\frac{1}{n^3}\right)\left(\sqrt{\frac{3n^4}{n^4}-\frac{n}{n^4}+\frac{7}{n^4}}+\sqrt{\frac{3n^4}{n^4}+\frac{7n^3}{n^4}-\frac{12}{n^4}}\right)}$$

$$= \frac{\left(-\frac{1}{n^2}+\frac{19}{n^3}-7\right)\left(\sqrt{6+\frac{80}{n}}+\sqrt{6-\frac{1}{n^4}}\right)}{\left(80+\frac{1}{n^3}\right)\left(\sqrt{3-\frac{1}{n^3}+\frac{7}{n^4}}\sqrt{3+\frac{7}{n}-\frac{12}{n^4}}\right)}$$

$$\xrightarrow{n\to\infty} \frac{(0+0-7)\left(\sqrt{6+0}+\sqrt{6-0}\right)}{(80+0)\left(\sqrt{3-0+0}\sqrt{3+0-0}\right)} = \frac{-7\cdot 3+3}{80(\sqrt{3}\sqrt{3}}$$

$$\cancel{= \frac{-7\cdot 2\sqrt{6}}{}} = \frac{-7\cdot\cancel{2}\sqrt{6}}{80\cdot\cancel{2}\sqrt{3}} = \frac{-7\cdot\sqrt{6}}{80\cdot\sqrt{3}} = \frac{-7}{80}\sqrt{2}$$

Vollständiges Video:

https://www.youtube.com/watch?v=XvccrQ1CGp0

$$\lim_{x\to\infty} \frac{\ln(x^4+12x^2-x)}{\sqrt{x}}$$

$$= \frac{\ln(x^4+12x^2-x)}{\sqrt{x}}$$

$$= \frac{\ln\left(x^4\left(1+\frac{12x^2}{x^4}-\frac{x}{x^4}\right)\right)}{\sqrt{x}}$$

$$= \frac{\ln(x^4)+\ln\left(1+\frac{12x^2}{x^4}-\frac{x}{x^4}\right)}{\sqrt{x}}$$

$$= \frac{4\ln(x)+\ln\left(1+\frac{12}{x^2}-\frac{1}{x^3}\right)}{\sqrt{x}}$$

$$= \underbrace{\frac{4\ln(x)}{\sqrt{x}=x^{\frac{1}{2}}}}_{0} + \frac{\ln\left(1+\frac{12}{x^2}-\frac{1}{x^3}\right)}{\sqrt{x}=x^{\frac{1}{2}}}$$

$$\xrightarrow[x\to\infty]{} 4\cdot 0+0=\underline{\underline{0}}$$

$$\lim_{x \to \infty} \frac{2}{x^{2n}+2}$$

Fall 1: $|x| < 1$

$$\lim_{x \to \infty} \frac{2}{x^{2n}+2} = \frac{2}{2} = 1$$

Fall 2: $x = 1$

$$\lim_{n \to \infty} \frac{2}{x^{2n}+2} = \frac{2}{1+2} = \frac{2}{3}$$

Fall 3: $x > 1$

$$\frac{2}{x^{2n}+2} = 0$$

$$\Rightarrow f(x) = \begin{cases} 1 & |x| < 1 \\ \frac{2}{3} & x = 1 \\ 0 & x > 1 \end{cases}$$

Fall 4: $x \leq -1$

unbestimmt divergent!

Vollständiges Video:

https://www.youtube.com/watch?v=TTK5N4VCtlI

$$\lim_{x \to \infty} \frac{e^{x^3}}{(2x)^{2x}}$$

$$\frac{e^{x^3}}{(2x)^{2x}} = e^{\ln\left(\frac{e^{x^3}}{(2x)^{2x}}\right)}$$

$$= e^{\ln(e^{x^3}) - \ln(2x)^{2x}}$$

$$= e^{x^3 \cancel{\ln(e)} - 2x \ln(2x)}$$

$$= e^{x^3 - 2x \ln(2x)}$$

$$= e^{x^2 (x - \frac{2 \ln(2x)}{x}}$$

$$= e^{x^2 (x - 2 \frac{\ln(2x)}{2x} \cdot 2x}$$

$$= e^{x^2 (x - 4 \frac{\ln(2x)}{2x}}$$

$$\xrightarrow[x \to \infty]{} e^{\infty(\infty - (4 \cdot 0))} = e^{\infty} = \infty$$

$$\lim_{n\to\infty} \sqrt{n+\sqrt{n}} - \sqrt{n}$$

$a^2 - b^2 = (a-b)(a+b)$

Vollständiges Video:

https://www.youtube.com/watch?v=8HBwHFybz08

$$\left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right)$$

$$= \frac{\left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right) \cdot \left(\sqrt{n+\sqrt{n}} + \sqrt{n}\right)}{\left(\sqrt{n+\sqrt{n}} + \sqrt{n}\right)}$$

$$= \frac{\left(\sqrt{n+\sqrt{n}}\right)^2 - \left(\sqrt{n}\right)^2 \cdot \left(\sqrt{n+\sqrt{n}} + \sqrt{n}\right)}{\left(\sqrt{n+\sqrt{n}} + \sqrt{n}\right)}$$

$$= \frac{\left(\cancel{n} + \sqrt{n} - \cancel{n}\right)}{\left(\sqrt{n+\sqrt{n}} + \sqrt{n}\right)} = \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}} + \sqrt{n}} \cdot$$

$$\boxed{\frac{\sqrt{n}}{n} = \frac{n^{\frac{1}{2}}}{n} = n^{\frac{1}{2}-1} = n^{-\frac{1}{2}} = \frac{1}{n^{\frac{1}{2}}} = \frac{1}{\sqrt{n}}}$$

$$= \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\sqrt{\frac{n}{n} + \frac{\sqrt{n}}{n}} + \frac{\sqrt{n}}{\sqrt{n}}} = \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}} + 1}$$

$$\xrightarrow{n\to\infty} \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{\ln(a^n+b^n)}{n} \qquad a>b>1$$

$$\frac{\ln(a^n+b^n)}{n} = \frac{\ln\left(a^n\cdot\left(1+\frac{b^n}{a^n}\right)\right)}{n}$$

Vollständiges Video:

https://www.youtube.com/watch?v=uS1AUaCgjGA

$$= \frac{\ln(a^n) + \ln\left(1+\frac{b^n}{a^n}\right)}{n}$$

$$= \frac{\cancel{n}\ln(a)}{\cancel{n}} + \frac{\ln\left(1+\frac{b^n}{a^n}\right)}{n}$$

$$= \ln(a) + \frac{\ln\left(1+\frac{b^n}{a^n}\right)}{n}$$

$$= \ln(a) + \frac{\ln\left(1+\overbrace{\left(\frac{b}{a}\right)^n}^{x^n}\right)}{n}$$

$$\xrightarrow[n\to\infty]{} \ln(a) + 0 = \underline{\underline{\ln(a)}}$$

$a>b>1$

$\frac{b}{a}$
$\underbrace{}_{\text{größer}}$

$\frac{z}{z^2} = 0,0\ldots$

$\Rightarrow z<1$

$$\lim_{n\to\infty} \left(\sqrt{\frac{4+n}{n}}\right)^{3n}$$

$$\left(\sqrt{\frac{4+n}{n}}\right)^{3n} = \left(\frac{4+n}{n}\right)^{\frac{1}{2}\cdot 3n}$$

$$= \left(\frac{4}{n}+\frac{n}{n}\right)^{\frac{1}{2}\cdot 3n} = \left(\frac{4}{n}+1\right)^{\frac{1}{2}\cdot 3n}$$

$$= \Big(\underbrace{1+\frac{4}{n}}_{e^4}\Big)^{\frac{1}{2}\cdot 3n} \xrightarrow{n\to\infty} \left(e^{4}\right)^{\frac{1}{2}\cdot 3}$$

Formelsammlung

$$= \left(e^{4}\right)^{\frac{3}{2}} = e^{4\cdot\frac{3}{2}} = e^{\frac{12}{2}}$$

$$= e^{\frac{6}{1}} = \underline{\underline{e^{6}}}$$

1 Variante

Vollständiges Video:
https://www.youtube.com/watch?v=R6rt5Ns6PSM

$$\lim_{n \to \infty} \ln(n) - \sqrt{n}$$

$$\ln(n) - \sqrt{n}$$

$$= \sqrt{n}\left(\frac{\ln(n)}{\sqrt{n}} - 1\right)$$

$$\xrightarrow[n \to \infty]{} \infty \cdot (-1) = \underline{\underline{-\infty}}$$

$$\lim_{n\to\infty} (\ln(n) - \sqrt{n})$$

(2. Variante)

Vollständiges Video:
https://www.youtube.com/watch?v=3ERQ5m8zCz0

$$(\ln(n) - \sqrt{n})$$

$$= \frac{(\ln(n) - \sqrt{n}) \cdot (\ln(n) + \sqrt{n})}{\ln(n) + \sqrt{n}}$$

$$= \frac{\ln(n)^2 - n}{\ln(n) + \sqrt{n}} = \frac{2\ln(n) - n}{\ln(n) + \sqrt{n}}$$

$$= \frac{2 \cdot \frac{\ln(n)}{n} - \frac{n}{n}}{\frac{\ln(n)}{n} + \frac{\sqrt{n}}{n}}$$

NR:

$$\frac{\sqrt{n}}{n} = \frac{n^{\frac{1}{2}}}{n^1} = n^{\frac{1}{2}-1} = n^{-\frac{1}{2}} = \frac{1}{n^{\frac{1}{2}}} = \frac{1}{\sqrt{n}}$$

$$\frac{1}{\infty} = 0$$

$$\frac{1}{0} = \infty$$

$$= \frac{2 \cdot \frac{\ln(n)}{n} - 1}{\frac{\ln(n)}{n} + \frac{1}{\sqrt{n}}}$$

$$\xrightarrow[n\to\infty]{} \frac{2 \cdot 0 - 1}{0 + 0} = \frac{-1}{0}$$

$$= -\infty$$

$$\lim_{n\to\infty} \frac{\ln(3n^n) - \ln((n+1)^n)}{\ln(2^n) - n\cdot\ln((n+1)}$$

Vollständiges Video:

https://www.youtube.com/watch?v=OtjTm0XNKX4

$$\frac{\ln(3n^n) - \ln((n+1)^n)}{\ln(2^n) - n\cdot\ln((n+1))}$$

$$= \frac{\ln(3) + \cancel{n}\ln(n^n) - n\ln((n+1))}{n\ln(2) - n\cdot\ln((n+1))}$$

$$= \frac{\ln(3) + n\ln(n) - n\ln(n+1)}{n\ln(2) - n\cdot\ln((n+1))}$$

$$= \frac{\cancel{n}\left(\frac{\ln(3)}{n} + \ln(n) - \ln(n+1)\right)}{\cancel{n}(\ln(2) - \ln((n+1))}$$

$$= \frac{\frac{\ln(3)}{n} + \ln\left(\frac{n}{n+1}\right)}{\ln(2) - \ln((n+1))} = \frac{\frac{\ln(3)}{n} + \ln\left(\frac{n}{n+1}\right.}{\ln\left(\frac{2}{n+1}\right)}$$

$$= \frac{\frac{\ln(3)}{n} + \ln\left(\frac{n+1}{n+1} - \frac{1}{n+1}\right)}{\ln\left(\frac{2}{n+1}\right)}$$

$\ln(1-0)$

$\ln(1) = 0$

$$\xrightarrow[n\to\infty]{} \frac{0+0}{-\infty} = 0 \checkmark$$

Vollständiges Video:

https://www.youtube.com/watch?v=vxzxrc7br28

$$\lim_{x\to 0^+} \sqrt{\frac{1}{x}+1}\left(\sqrt{\frac{1}{x}+a}-\sqrt{\frac{1}{x}+1}\right)$$

$$\sqrt{\frac{1}{x}+1}\left(\sqrt{\frac{1}{x}+a}-\sqrt{\frac{1}{x}+1}\right)$$

$$=\sqrt{\frac{1}{x}+1}\,\frac{\left(\sqrt{\frac{1}{x}+a}-\sqrt{\frac{1}{x}+1}\right)\left(\sqrt{\frac{1}{x}+a}+\sqrt{\frac{1}{x}+1}\right)}{\left(\sqrt{\frac{1}{x}+a}+\sqrt{\frac{1}{x}+1}\right)}$$

$$=\sqrt{\frac{1}{x}+1}\,\frac{\left(\sqrt{\frac{1}{x}+a}\right)^2-\left(\sqrt{\frac{1}{x}+1}\right)^2}{\left(\sqrt{\frac{1}{x}+a}+\sqrt{\frac{1}{x}+1}\right)}=\frac{\sqrt{\frac{1}{x}+1}\left(\cancel{\frac{1}{x}}+a-\cancel{\frac{1}{x}}-1\right)}{\left(\sqrt{\frac{1}{x}+a}+\sqrt{\frac{1}{x}+1}\right)}$$

$$=\frac{\sqrt{\frac{1}{x}+1}\,(a-1)}{\left(\sqrt{\frac{1}{x}+a}+\sqrt{\frac{1}{x}+1}\right)}=\frac{\sqrt{\frac{1}{x}+\frac{x}{x}}\,(a-1)}{\left(\sqrt{\frac{1}{x}+a}+\left(\sqrt{\frac{1}{x}+1}\right.\right.}$$

$$=\frac{\sqrt{\frac{1+x}{x}}\,(a-1)}{\left(\sqrt{\frac{1}{x}+a}+\left(\sqrt{\frac{1}{x}+1}\right)\right.}\;\cdot\sqrt{x}$$

$$=\frac{\sqrt{\frac{x+x^2}{x}}\,(a-1)}{\left(\sqrt{\frac{\cancel{x}}{x}+ax}+\left(\sqrt{\frac{x}{x}+x}\right.\right.}=\frac{\sqrt{1+x}\,(a-1)}{\left(\sqrt{1+ax}+\left(\sqrt{1+x}\right.\right.}$$

$$\xrightarrow[x\to 0^+]{}\quad\frac{\sqrt{1+0}\,(a-1)}{\sqrt{1+0}+\left(\sqrt{1+0}\right.}=\frac{a-1}{2}$$

$$\lim_{x \to \infty} \left(\frac{n^2+3n+2}{n^2+4n-5}\right)^n$$

Linearfaktorzerlegung:

$$\left(\frac{n^2+3n+2}{n^2+4n-5}\right)^n$$

Vollständiges Video:

https://www.youtube.com/watch?v=rqy9UeKIz1c

$$= \left(\frac{(n+1)(n+2)}{(n-1)(n+5)}\right)^n$$

$$= \left(\frac{\cancel{n}\left(1+\frac{1}{n}\right) \cancel{n}\left(1+\frac{2}{n}\right)}{\cancel{n}\left(1-\frac{1}{n}\right) \cancel{n}\left(1+\frac{5}{n}\right)}\right)^n$$

$$= \left(\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{\left(1-\frac{1}{n}\right)\left(1+\frac{5}{n}\right)}\right)^n$$

$$= \frac{\left(1+\frac{1}{n}\right)^n \left(1+\frac{2}{n}\right)^n}{\left(1-\frac{1}{n}\right)^n \left(1+\frac{5}{n}\right)^n}$$

$$= \frac{e^1 \cdot e^2}{e^{-1} \cdot e^5} = \frac{e^3}{e^4} = e^{3-4} = e^{-1} = \frac{1}{e}$$

n^2+3n+2

$$-\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - \frac{2}{1}}$$

$$-\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}$$

$$-\frac{3}{2} \pm \sqrt{\frac{1}{4}}$$

$$-\frac{3}{2} + \frac{1}{2} = -\frac{2}{2} = \boxed{-1}$$

$$-\frac{3}{2} - \frac{1}{2} = -\frac{4}{2} = \boxed{-2}$$

$$-\frac{4}{2} \pm \sqrt{\left(\frac{4}{2}\right)^2 + 5}$$

$$-2 \pm \sqrt{4+5}$$

$$-2 + 3 = \boxed{1}$$

$$-2 - 3 = \boxed{-5}$$

$$\lim_{x\to\infty} \arctan(\ln(x) - \sqrt{x})$$

$$\arctan(\ln(x) - \sqrt{x})$$

$$= \arctan\left(\sqrt{x}\left(\frac{\ln(x)}{\sqrt{x}} - 1\right)\right)$$

$$\xrightarrow{x\to\infty} \arctan(\infty\,(0-1))$$

$$= \arctan(-\infty) = -\frac{\pi}{2}$$

$$\lim_{n\to\infty} \frac{\ln(4n^n) - \ln(\sqrt[5]{n})}{\ln(5n^{2n}) - \ln(\sqrt[5]{n})}$$

$$\frac{\ln(4n^n) - \ln(\sqrt[5]{n})}{\ln(5n^{2n}) - \ln(\sqrt[5]{n})} = \frac{\ln(4n^n) - \ln(n^{\frac{1}{5}})}{\ln(5n^{2n}) - \ln(n^{\frac{1}{5}})}$$

$$= \frac{\ln(4) + \ln(n^n) - \ln(n^{\frac{1}{5}})}{\ln(5) + \ln(n^{2n}) - \ln(n^{\frac{1}{5}})}$$

$$= \frac{\ln(4) + \ln\left(\frac{n^n}{n^{\frac{1}{5}}}\right)}{\ln(5) + \ln\left(\frac{n^{2n}}{n^{\frac{1}{5}}}\right)} = \frac{\ln(4) + \ln(n^{n-\frac{1}{5}})}{\ln(5) + \ln(n^{2n-\frac{1}{5}})}$$

$$= \frac{\ln(4)}{\ln(5) + \ln(n^{2n-\frac{1}{5}})} + \frac{\ln(n^{n-\frac{1}{5}})}{\ln(5) + \ln(n^{2n-\frac{1}{5}})} \quad |:n$$

$$= \frac{\ln(4)}{\ln(5) + \ln(n^{2n-\frac{1}{5}})} + \frac{\left(n - \frac{1}{5}\right) \cdot \ln(n)}{\ln(5) + \cancel{\ln}\, 2n - \frac{1}{5} \cdot \ln(n)}$$

$$\boxed{\frac{\frac{1}{5}}{\frac{n}{1}} = \frac{1}{5n}}$$

$$= \frac{\ln(4)}{\ln(5) + \ln(n^{2n-\frac{1}{5}})} + \frac{\left(\frac{n}{n} - \frac{1}{5n}\right) \cdot \ln(n)}{\frac{\ln(5)}{n} + \frac{2n}{n} - \frac{1}{5n} \cdot \ln(n)}$$

$$= \frac{\ln(4)}{\ln(5) + \ln(n^{2n-\frac{1}{5}})} + \frac{\left(1 - \frac{1}{5n}\right) \cdot \ln(n)}{\frac{\ln(5)}{n} + \left(2 - \frac{1}{5n}\right) \cdot \ln(n)}$$

$$= \frac{\ln(4)}{\ln(5) + \ln\left(n^{2n-\frac{1}{5}}\right)} + \frac{\left(1-\frac{1}{5n}\right) \cdot \frac{\ln(n)}{\ln(n)}}{\frac{\frac{\ln(5)}{n}}{\ln(n)} + \left(2-\frac{1}{5n}\right) \cdot \frac{\ln(n)}{\ln(n)}}$$

$$= \frac{\ln(4)}{\ln(5) + \ln\left(n^{2n-\frac{1}{5}}\right)} + \frac{\left(1-\frac{1}{5n}\right) \cdot 1}{\frac{\ln(5)}{\ln(n)^{n}} + 2-\frac{1}{5n} \cdot 1}$$

$$\xrightarrow[n\to\infty]{} 0 + \frac{(1-0) \cdot 1}{0 + 2 - 0 \cdot 1} = \frac{1}{2}$$

Randverhalten Tutorium

$$f(x) = \frac{x(x-3)\ \ln(|x|)}{(x-4)(x+5)\ \ln(|x^2-3|)}$$

1) Df

$\ln(|x|) \neq 0$ | $x-4 \neq 0$ | $x+5 \neq 0$

$\boxed{x \neq 0}$ | $\boxed{x \neq 4}$ | $\boxed{x \neq -5}$

$\ln(|x^2-3|) \neq 0$

$|x^2-3| = 1$

$x^2-3 \neq 1$

$x^2 \neq 4$

$|x| \neq \sqrt{4}$

$\boxed{x \neq 2}$

$-x \neq 2$

$\boxed{x \neq -2}$

$-x^2+3 \neq 1$

$-x^2 \neq -2$

$x^2 \neq 2$

$\boxed{x \neq \sqrt{2}}$

$-x \neq \sqrt{2}$

$\boxed{x \neq -\sqrt{2}}$

$|x^2-3| \neq 0$

$x^2 \neq 3$

$\boxed{x \neq \sqrt{3}}$ $\boxed{x \neq -\sqrt{3}}$

$Df = \mathbb{R} \setminus \{0, 4, -5, 2, -2, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}\}$

✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

$D_f = \mathbb{R} \setminus \{0, 4, -5, 2, -2, \sqrt{2}, \sqrt{2}, \sqrt{3}, -\sqrt{3}\}$

$$f(x) = \frac{(x^2-3x)\ \ln(|x|)}{(x-4)(x+5)\ln(|x^2-3|)}$$

$\lim\limits_{x\to 4^+} f(x) = \infty$ $\quad \frac{++}{+++}$	$\lim\limits_{x\to 4^-} f(x) = -\infty$ $\quad \frac{++}{-++}$
$\lim\limits_{x\to -5^+} f(x) = -\infty$ $\quad \frac{++}{-++}$	$\lim\limits_{x\to -5^-} f(x) = \infty$ $\quad \frac{++}{--+}$
$\lim\limits_{x\to 2^+} f(x) = +\infty$ $\quad \frac{-+}{-++}$	$\lim\limits_{x\to 2^-} f(x) = -\infty$ $\quad \frac{-+}{-+-}$
$\lim\limits_{x\to -2^+} f(x) = +\infty$ $\quad \frac{++}{-+-}$	$\lim\limits_{x\to -2^-} f(x) = -\infty$ $\quad \frac{++}{-++}$
$\lim\limits_{x\to \sqrt{2}^+} f(x) = -\infty$ $\quad \frac{-+}{-+-}$	$\lim\limits_{x\to \sqrt{2}^-} f(x) = +\infty$ $\quad \frac{-+}{-++}$
$\lim\limits_{x\to -\sqrt{2}^+} f(x) = -\infty$ $\quad \frac{++}{-++}$	$\lim\limits_{x\to -\sqrt{2}^-} f(x) = +\infty$ $\quad \frac{++}{-+-}$
$\lim\limits_{x\to -\sqrt{3}^+} = 0$	$\lim\limits_{x\to -\sqrt{3}^-} = 0$
$\lim\limits_{x\to +\sqrt{3}^+} = 0$	$\lim\limits_{x\to \sqrt{3}^-}\ 0$

$$\lim_{x\to\infty} \frac{(x^2-3x)\ \ln(|x|)}{(x-4)(x+5)\ln(|x^2-3|)}$$

$$\frac{(x^2-3x)\ \ln(|x|)}{(x-4)(x+5)\ln(|x^2-3|)}$$

$$= \frac{(x^2-3x)\ \ln(|x|)}{(x-4)(x+5)\ln\left(\left|x^2\left(1-\frac{3}{x^2}\right)\right|\right)} = \frac{(x^2-3x)\ln(|x|)}{(x-4)(x+5)\ln(|x|)^2+\ln\left(\left|1-\frac{3}{x^2}\right|\right)}$$

$$= \frac{(x^2-3x)\ \ln(|x|)}{(x-4)(x+5)\,2\ln(|x|)+\ln\left(\left|1-\frac{3}{x^2}\right|\right)} = \frac{x(x-3)\frac{\ln(|x|)}{\ln(|x|)}}{(x-4)(x+5)\,2\frac{\ln(|x|)}{\ln(|x|)}+\frac{\ln\left(\left|1-\frac{3}{x^2}\right|\right)}{\ln(|x|)}}$$

$$= \cancel{(x-3)\,x\ln(|x|)} = \frac{(x^2-3x)}{(x-4)(x+5)\,2+\frac{\ln\left(\left|1-\frac{3}{x^2}\right|\right)}{\ln(|x|)}}$$

$$= \frac{\left(\frac{x^2}{x^2}-\frac{3x}{x^2}\right)}{\left(\frac{x}{x}-\frac{4}{x}\right)\left(\frac{x}{x}+\frac{5}{x}\right)2+\frac{\ln\left(\left|1-\frac{3}{x^2}\right|\right)}{\ln(|x|)}} = \frac{\left(1-\frac{3}{x}\right)}{\left(1-\frac{4}{x}\right)\left(1+\frac{5}{x}\right)2+\frac{\ln\left(\left|1-\frac{3}{x^2}\right|\right)}{\ln(|x|)}}$$

$$\xrightarrow[x\to\infty]{} \frac{1}{(1-0)(1+0)\,2+0} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{1}{2}$$

$$\lim_{x\to 0} \frac{x(x-3)\ \ln(|x|)}{(x-4)(x+5)\ln(|x^2-3|)} .$$

$$\frac{(x-3)\cdot \overbrace{x\cdot\ln(|x|)}^{0}}{(x-4)(x+5)\ln(|x^2-3|)} \xrightarrow[x\to 0]{} \frac{0}{\text{Zahl}} = \underline{\underline{0}}$$

Randverhalten Tutorium

$$f(x) = \frac{(x^3-3x^2+3x-1)\ln|x|}{(x^3-x^2+x-1)\ln|x-e|}$$

Df?

$$(x^3-3x^2+3x-1):(x-1) = x^2-2x+1$$
$$-(x^3-x^2)$$
$$-2x^2+3x$$
$$-(-2x^2+2x)$$
$$x-1$$
$$-(x-1)$$
$$0$$

$$(x^3-x^2+x-1):(x-1) = x^2+1$$
$$-(x^3-x^2)$$
$$x-1$$
$$-(x-1)$$
$$0$$

$$\Rightarrow f(x) = \frac{(x^2-2x+1)(x-1)\ln|x|}{(x^2+1)(x-1)\ln|x-e|}$$

$x^2+1 \neq 0$
$x^2 \neq -1$
↯

$x-1 \neq 0$
$\boxed{x \neq 1}$

$\ln|x| \neq 0$
$\boxed{x \neq 0}$

$\ln|x-e| \neq 0$
$|x-e| \neq 1$
$x-e \neq 1 \quad | \; -x+e \neq 1$
$\boxed{x \neq 1+e} \quad | \; -x \neq 1-e$
$\boxed{x \neq -1+e}$

Df: $\mathbb{R} \setminus \{0, 1, 1+e, -1+e, e\}$

$|x-e| \neq 0$

$-x+e \neq 0$
$-x \neq -e$
$\boxed{x \neq e}$

$x-e \neq 0$
$\boxed{x \neq e}$

$D_f = \mathbb{R} \setminus \{1, 0, 1+e, -1+e, e\}$

$$f(x) = \frac{(x^2-2x+1)\,\cancel{(x-1)}\,\ln|x|}{(x^2+1)\,\cancel{(x-1)}\,\ln|x-e|}$$ Punktlücke bei 1

$\lim\limits_{x\to 1^+} f(x) \underset{\frac{(1-2+1)\,2\alpha}{(2}}{=} 0$	$\lim\limits_{x\to 1^-} f(x) = 0$
$\lim\limits_{x\to 1+e^+} f(x) = \infty \quad \frac{++}{++}$	$\lim\limits_{x\to 1+e^-} f(x) = -\infty \quad \frac{++}{+-}$
$\lim\limits_{x\to -1+e^+} f(x) = -\infty \quad \frac{++}{+-}$	$\lim\limits_{x\to -1+e^-} f(x) = +\infty \quad \frac{++}{++}$
$\lim\limits_{x\to e^+} f(x) = 0 \quad \frac{0}{-\infty}$	$\lim\limits_{x\to e^-} f(x) = 0 \quad \frac{0}{-\infty}$
$\lim\limits_{x\to 0^+} f(x) = -\infty$	$\lim\limits_{x\to 0^-} f(x) = -\infty$

$$\lim_{x\to\infty} f(x) = \frac{(x^2-2x+1)\ln|x|}{(x^2+1)\ln|x-e|}$$

$$= \frac{(x^2-2x+1)\ln|x|}{(x^2+1)\ln|x-e|} = \frac{(x^2-2x+1)\ln|x|}{(x^2+1)\ln(x)+\ln(1-\frac{e}{x})}$$

$$= \frac{(x^2-2x+1)\frac{\ln|x|}{\ln|x|}}{(x^2+1)\frac{\ln|x|}{\ln|x|}+\frac{\ln(1-\frac{e}{x})}{\ln|x|}} = \frac{(x^2-2x+1)}{(x^2+1)\cdot 1+\frac{\ln(1-\frac{e}{x})}{\ln|x|}} \quad \Big|:x^2$$

$$= \frac{\left(1-\frac{2}{x}+\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)1+\frac{\ln(1-\frac{e}{x})}{\ln(|x|)}} \xrightarrow{x\to\infty} \frac{(1-0+0)}{(1+0)\,1+0} = 1$$

$$\lim_{x\to-\infty} 1$$

Randverhalten Tutorium

$$f(x) = \frac{x^3 - 1}{(x^2-1)(2x+5)e^{\frac{1}{x}}}$$

$$(x^3-1) : (x-1) = x^2+x+1$$
$$\frac{-(x^3-x^2)}{x^2-1}$$
$$\frac{-(x^2-x)}{x-1}$$
$$\frac{-(x-1)}{0}$$

$$\Rightarrow \frac{(x^2+x+1)(x-1)}{(x-1)(x+1)(2x+5)e^{\frac{1}{x}}}$$

iQE: (x^2+x+1)

$$\Leftrightarrow \left(x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1\right)$$
$$\Leftrightarrow \left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1$$
$$\Leftrightarrow \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

(Unnötig)
(nur Übung)

Df) $\boxed{x \neq -1}$
$\boxed{x \neq 1}$

$$2x+5 \neq 0$$
$$2x \neq -5$$
$$\boxed{x \neq -\frac{5}{2}}$$

$$e^{\frac{1}{x}} \neq 0$$
$$\frac{1}{x} \neq 0$$
$$\boxed{x \neq 0}$$

$$Df = \mathbb{R} \setminus \{-1, 1, -\frac{5}{2}, 0\}$$

$$D_f = \mathbb{R} \setminus \{-1, 1, -\tfrac{5}{2}, 0\}$$

$$f(x) = \frac{(x^2+x+1)\ \cancel{(x-1)}}{\cancel{(x-1)}(x+1)(2x+5)e^{\frac{1}{x}}}$$

$$= \frac{(x^2+x+1)}{(x+1)(2x+5)e^{\frac{1}{x}}}$$

$\Rightarrow$ bei 1 muss Punktlücke

$$\lim_{x\to 1} f(x) = \frac{3}{14e}$$

$$\lim_{x\to -1^+} f(x) = +\infty \quad \frac{+}{+++}$$

$$\lim_{x\to -1^-} f(x) = -\infty \quad \frac{+}{-++}$$

$$\lim_{x\to -\frac{5}{2}^+} f(x) = -\infty \quad \frac{+}{-++}$$

$$\lim_{x\to -\frac{5}{2}^-} f(x) = +\infty \quad \frac{+}{--+}$$

$$\lim_{x\to 0^+} f(x) = 0 \quad \frac{1}{e^{\infty}}$$

$$\lim_{x\to 0^-} e^{-\infty} = \frac{1}{0} = \infty$$

$$\lim_{x\to\infty} \frac{\left(\frac{x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}\right)}{\left(\frac{x}{x}+\frac{1}{x}\right)\cdot\left(\frac{2x}{x}+\frac{5}{x}\right)\cdot e^{\frac{1}{x}}} \xrightarrow[x\to\infty]{} \frac{1+0+0}{(1+0)\cdot(2+0)\cdot 1} = \frac{1}{2}$$

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}$$

$$= \lim_{x\to\infty} \frac{(1+1/x+1/x^2)}{(1+1/x)(2+5/x)\,e^{1/x}}$$

Randverhalten Tutorium

$$f(x) = \frac{(x^2-6x-16)\ \ln(|x|+2)}{(x^2-11x+24)\ \ln|x|}$$

Df)

$(x^2-11x+24)$

$x^2-11x+\left(\frac{11}{2}\right)^2-\left(\frac{11}{2}\right)^2+24$

$\left(x-\frac{11}{2}\right)^2-\frac{121}{4}+24$

$-\frac{121}{4}+\frac{96}{4}$

$\left(x-\frac{11}{2}\right)^2-\frac{25}{4} \neq 0$

$\left(x-\frac{11}{2}\right)^2 \neq \frac{25}{4}$

$\left|x-\frac{11}{2}\right| \neq \frac{5}{2}$

$x-\frac{11}{2} \neq \frac{5}{2}$

$x \neq \frac{5}{2}+\frac{11}{2}$

$x \neq \frac{16}{2}$

$\boxed{x \neq 8}$

$-x+\frac{11}{2} \neq \frac{5}{2}$

$-x \neq \frac{5}{2}-\frac{11}{2}$

$-x \neq \frac{5}{2}-\frac{11}{2}$

$-x \neq \frac{-6}{2}$

$-x = -3$

$\boxed{x = 3}$

$x^2-6x-16$

$x^2-6x+\left(\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2-16$

$(x-3)^2-9-16$

$(x-3)^2-25 \neq 0$

$(x-3)^2=25$

$(x-3)=\sqrt{25}$

$-x+3 \neq 5$

$-x \neq 2$

$\boxed{x \neq -2}$

$x-3=5$

$\boxed{x = 8}$

$\ln|x|$

$\boxed{x \neq 1}$

$\boxed{x \neq -1}$

$\boxed{x \neq 0}$

$Df = \mathbb{R} \setminus \{8, 3, 1, -1, 0\}$

$$D_f = \mathbb{R} \setminus \{8, 3, 1, -1, 0\}$$

$$f(x) = \frac{(\cancel{x-8})(x+2)\ \ln(|x|+2)}{(\cancel{x-8})(x-3)\ \ln(|x|)}$$

Punktlücke bei 8

$$f(x) = \frac{(x+2)\ \ln(|x|+2)}{(x-3)\ \ln(|x|)}$$

$$\lim_{x\to 8^+} f(x) = \frac{10\ln(10)}{5\ln(8)} = \frac{2\ln(10)}{1\ \ln(8)} = \frac{\ln 10^2}{\ln 8^1} = \frac{\ln(100)}{\ln(8)}$$

$$\lim_{x\to 3^+} f(x) = +\infty \quad \frac{++}{+\cdot+}$$ | $$\lim_{x\to 3^-} f(x) = -\infty \quad \frac{++}{-+}$$

$$\lim_{x\to 1^+} f(x) = -\infty \quad \frac{++}{-+}$$ | $$\lim_{x\to 1^-} f(x) = +\infty \quad \frac{++}{--}$$

$$\lim_{x\to -1^+} f(x) = \infty \quad \frac{++}{--}$$ | $$\lim_{x\to -1^-} f(x) = -\infty \quad \frac{++}{-+}$$

$$\lim_{x\to 0^+} f(x) = 0 \quad \frac{2\ln(2)}{-1\ln(0)}$$ | $$\lim_{x\to 0^-} f(x) = 0$$

$$\lim_{x\to\infty} \frac{\left(\frac{x}{x}+\frac{2}{x}\right)\ln\left(x\left(1+\frac{2}{x}\right)\right)}{\left(\frac{x}{x}-\frac{3}{x}\right)\ln(x)} = \frac{\left(1+\frac{2}{x}\right)\left(\ln(x)+\ln\left(1+\frac{2}{x}\right)\right)}{\left(1-\frac{3}{x}\right)\ \ln(x)}$$

$$= \frac{\left(1+\frac{2}{x}\right)\left(\frac{\ln(|x|)}{\ln(|x|)}^{1} + \frac{\ln\left(1+\frac{2}{x}\right)}{\ln(x)}\right)}{\left(1-\frac{3}{x}\right)\ \frac{\ln x}{\ln(x)}} \xrightarrow[x\to\infty]{} \frac{(1+0)\cdot 1+0}{(1-0)\cdot 1} = \frac{1}{1}$$

1

Randverhalten Tutorium

$$f(x) = \frac{(x+2)(x+1)}{(x-2)(x^2-1)\ln(x^2+1)}$$

$Df)$ $\boxed{x \neq 2}$ $\quad x^2-1 \neq 0$, $x^2 \neq 1$, $\boxed{x \neq 1}$, $-x \neq 1$, $\boxed{x \neq -1}$

$x^2+1 \neq 1$, $x^2 \neq 0$, $\boxed{x \neq 0}$ | $x^2+1 \neq 0$, $x^2 \neq -1$ ↯

~~$x^2+1 \neq 0$, $x^2 \neq 1$~~

$$Df = \mathbb{R} \setminus \{2, 1, -1, 0\}$$

$$\Rightarrow \frac{(x+2)(x+1)}{(x-2)(x^2-1)\ln(x^2+1)} = \frac{(x+2)\cancel{(x+1)}}{(x-2)(x-1)\cancel{(x+1)}\ln(x^2+1)}$$

-1 Punktlücke

$$= \frac{(x+2)}{(x-2)(x-1)\ln(x^2+1)} = f(x)$$

$$\lim_{x \to -1} f(x) = \frac{1}{(-3)(-2)\ln(2)} = \frac{1}{6(\ln 2)} = \frac{1}{\ln(6)^2} = \frac{1}{\ln(64)}$$

$$\lim_{x \to 1^+} f(x) = -\infty \quad \frac{+}{-++}$$

$$\lim_{x \to 1^-} f(x) = +\infty \quad \frac{+}{--+}$$

$$\lim_{x \to 2^+} f(x) = +\infty \quad \frac{+}{+++}$$

$$\lim_{x \to 2^-} f(x) = -\infty \quad \frac{+}{-++}$$

$$\lim_{x \to 0^+} f(x) = \infty \quad \frac{2}{(-2)(-1)\cdot 0} \quad \frac{+}{--+}$$

$$\lim_{x \to 0^-} f(x) = \infty$$

$$\lim_{x \to \infty} \frac{\left(\frac{x}{x^2}+\frac{2}{x^2}\right)}{\left(\frac{x}{x}-\frac{2}{x}\right)\left(\frac{x}{x}-\frac{1}{x}\right)\ln(x^2+1)} \xrightarrow{x \to \infty} \frac{(0+0)}{(1-0)(1-0)\ldots} = 0$$

$$\lim_{x \to -\infty} f(x) = 0$$

Stetigkeit Tutorium

$$f(x) = \lim_{n \to \infty} \frac{x^{n+1} + ax^2}{x^n + 2} \qquad x \in (-1, \infty)$$

Wähle den Parameter a, sodass f im Punkt $x = 1$ stetig ist.

dazu: $\frac{x^{n+1} + ax^2}{x^n + 2}$

Fall 1: $-1 < x < 1$, $|x| < 1$

$$\xrightarrow[n \to \infty]{} \frac{0 + ax^2}{0 + 2} = \frac{ax^2}{2}$$

Fall 2: $x > 1$

$$\frac{x^{n+1} + ax^2}{x^n + 2} = \frac{\frac{x^{n+1}}{x^{n+1}} + \frac{ax^2}{x^{n+1}}}{\frac{x^n}{x^{n+1}} + \frac{2}{x^{n+1}}} = \frac{1 + \frac{ax^2}{x^{n+1}}}{\frac{1}{x} + \frac{2}{x^{n+1}}}$$

$$\frac{x^n}{x^n \cdot x^1} \leftarrow \frac{x^n}{x^{n+1}}$$

$$\xrightarrow[n \to \infty]{} \frac{1 + 0}{\frac{1}{x} + 0} = \frac{1}{\frac{1}{x}} = \frac{\frac{1}{1}}{\frac{1}{x}} = \frac{1 \cdot x}{1 \cdot 1} = \frac{x}{1} = x$$

~~Stetigkeit~~ Fall 3: $x = 1$

$$\frac{1^{n+1} + a \cdot 1^2}{1^n + 2} \underset{n \to \infty}{=} \frac{1^1 + a}{1 + 2} = \frac{1 + a}{3}$$

Stetigkeit

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{ax^2}{2} = \frac{a}{2}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1$$

$$\frac{a}{2} = 1 = \frac{1+a}{3} = 2 = a$$

$a = 2$

$\Rightarrow \frac{a}{2} \stackrel{!}{=} 1$

$= a = 2$

$f(x)$ stetig in $x = 1$ wenn $a = 2$

Stetigkeit Tutorium

$$f(x) = \begin{cases} \frac{e^x + \alpha x - 1}{x} & , x < 0 \\ \ln(\beta x + 1) & , x \geq 0 ; \beta > 0 \end{cases}$$

lassen sich die Parameter α und β so bestimmen dass,

a) $\ln x = 0$ eine Punktlücke vorliegt.

$$\lim_{x \to 0^-} \frac{e^x + ax - 1}{x}$$

$$\frac{e^x + ax - 1}{x} = \frac{e^x - 1}{x} + \frac{a\cancel{x}}{\cancel{x}} = \frac{e^x - 1}{x} + a$$

$$\xrightarrow[x \to 0^-]{} 1 + \alpha$$

$$\lim_{x \to 0^+} \ln(\beta x + 1) = 0$$

a) $\alpha = -1$ Punktlücke

b) $\ln x = 0$ eine Sprungstelle mit s vorliegt

$\alpha = 4$, $\alpha = -6$

c) $\ln x = 0$ eine stetige Ergänzung möglich ist

$\alpha = -1$

Stetigkeit Tutorium

$$f(x) = \begin{cases} \frac{1-\cos x}{x} & \text{falls } x<0 \\ e^{-\frac{a}{x}} & \text{falls } x>0 \end{cases}$$

Überprüfe, ob Parameter $a \in \mathbb{R}$ so wählen lässt, dass im Punkt $x=0$ jeweils eine Punktlücke eine evtl. einseitiges Polverhalten oder eine Sprungstelle entsteht.

$!\ \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$2x = x$

$x = \frac{x}{2}$

$$\lim_{x\to 0^-} \frac{1-\cos x}{x}$$

$$\frac{1-\cos x}{x} = \frac{\frac{1}{2}(1-\cos x)}{x \cdot \frac{1}{2}} = \frac{(\sin^2 \frac{x}{2})}{x \cdot \frac{1}{2}}$$

$$= \frac{(\sin^2 \frac{x}{2})}{\frac{x}{2}} = \underbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}_{1} \cdot \underbrace{\sin\frac{x}{2}}_{0} \xrightarrow[x\to 0^-]{} 0$$

$$\lim_{x\to 0^+} e^{-\frac{a}{x}} \xrightarrow[x\to 0^+]{} \begin{cases} 0 & \text{wenn } a>0 & \text{Punktlücke} \\ \infty & \text{wenn } a<0 & \text{einseitiges Polv.} \\ 1 & \text{wenn } a=0 & \text{Sprungstelle} \end{cases}$$

Stetigkeit Tutorium

$$f(x) = \begin{cases} e^2 \cdot \dfrac{\left(\frac{x}{e}\right)^3 - 1}{\frac{x}{e} - 1} & x < e \\ \ln(ax^2) & x \geq e \end{cases}$$

den Parameter $\underline{a}$ so gewählt werden kann, dass f eine auf $\mathbb{R}$ stetige Funktion wird.

$$\lim_{x \to e^-} e^2 \cdot \frac{\left(\frac{x}{e}\right)^3 - 1}{\frac{x}{e} - 1}$$

$$\frac{e^2 \cdot \left(\frac{x}{e}\right)^3 - 1}{\frac{x}{e} - 1} = \frac{e^2 \cdot \frac{x^3}{e^3} - 1}{\frac{x}{e} - 1} = \frac{e^2 \cdot \frac{x^3}{e^3} - \frac{e^3}{e^3}}{\frac{x}{e} - \frac{e}{e}}$$

$$= e^2 \cdot \frac{\frac{x^3 - e^3}{e^3}}{\frac{x-e}{e}} = e^2 \frac{(x^3 - e^3) \cdot \cancel{e}}{e^{\cancel{3}2} \cdot (x-e)} = \frac{\cancel{e^2}(x^3 - e^3)}{\cancel{e^2}(x-e)}$$

$$= \frac{(x^3 - e^3)}{(x-e)}$$

$$(x^3 - e^3) : (x - e) = x^2 + xe + e^2$$

$$\begin{array}{l} -(x^3 - x^2 e) \\ \hline x^2 e - e^3 \\ -(x^2 e - x e^2) \\ \hline x e^2 - e^3 \\ -(x e^2 - e^3) \\ \hline \underline{\underline{0}} \end{array}$$

$$\Rightarrow x^2 + xe + e^2$$

$$\xrightarrow[x \to e^-]{} e^2 + e \cdot e + e^2$$

$$e^2 + e^2 + e^2$$

$$= \underline{\underline{3e^2}}$$

$$\lim_{x \to e^+} \ln(ax^2) = ?$$

$$\ln(ax^2) = \ln(a) + \ln(x^2) = \ln(a) + 2\ln(x)$$

$$\xrightarrow[x \to e^+]{} \ln a + 2 \cdot \ln(e) = \underline{\ln(a) + 2}$$

$$\Rightarrow \ln a + 2 = 3e^2$$

$$\ln a = 3e^2 - 2$$

$$\underline{\underline{a = e^{3e^2 - 2}}}$$

$a = e^{3e^2-2}$ muss gewählt werden für die Stetigkeit

Df / Umkehrfunktion / Symmetrie Tutorium teil 1

$$f(x) = \ln\left(\sqrt{\underbrace{\frac{ax+b}{b-ax}}_{>0}}\right) \quad a,b>0$$

a) Df ?

$$\frac{ax+b}{b-ax} > 0$$

1 Fall: $\frac{+}{+}$ $\quad ax+b > 0 \quad \wedge \quad b-ax > 0$

$\Leftrightarrow \quad ax > -b \qquad\qquad -ax > -b$

$\boxed{x > -\frac{b}{a}} \qquad\qquad \boxed{x < \frac{b}{a}} \qquad x \in \left(-\frac{b}{a}, \frac{b}{a}\right)$

2 Fall: $\frac{-}{-}$ $\quad ax+b < 0 \quad \wedge \quad b-ax < 0$

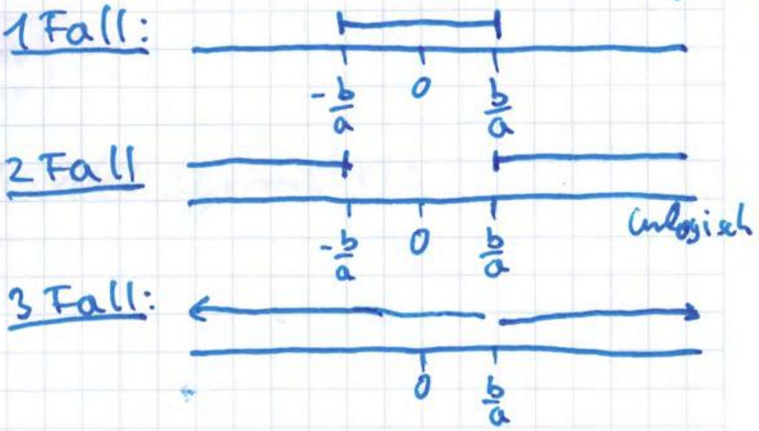
$\Leftrightarrow \quad ax < -b \qquad \Leftrightarrow \quad -ax < -b$

$\boxed{x < -\frac{b}{a}} \qquad\qquad \boxed{x > \frac{b}{a}} \qquad x \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$

3 Fall: $\quad b - ax \neq 0$

$-ax \neq -b$

$\boxed{x \neq \frac{b}{a}}$



~~xe~~

$$x \in \left(-\frac{b}{a}, \frac{b}{a}\right)$$

b) Symmetrie

Teil 2

$$f(x) = \ln\sqrt{\frac{ax+b}{b-ax}}$$

$$f(-x) = \ln\sqrt{\frac{a(-x)+b}{b+ax}}$$

$$= \ln\sqrt{\frac{-ax+b}{b+ax}}$$

$$= \ln\sqrt{\frac{b-ax}{\cancel{ax}+b}}$$

$$= \ln\sqrt{\frac{1}{\frac{ax+b}{b-ax}}}$$

$$= \ln\sqrt{\left(\frac{ax+b}{b-ax}\right)^{-1}}$$

$$= -\ln\sqrt{\left(\frac{ax+b}{b-ax}\right)}$$

$f(-x) = -f(x) \Rightarrow$ Punktsymmetrie

c) <u>Umkehrfunktion</u> Teil 3

$$y = \ln\left(\sqrt{\frac{ax+b}{b-ax}}\right)$$

$$\Leftrightarrow \quad e^{y} = \underbrace{\sqrt{\frac{ax+b}{b-ax}}}_{+}$$

$$\Leftrightarrow \quad e^{y^2} = \frac{ax+b}{b-ax}$$

$$\Leftrightarrow \quad e^{y^2} \cdot (b-ax) = ax+b$$

$$\Leftrightarrow (e^{y^2} \cdot b) - (e^{y^2} \cdot ax) = ax+b$$

$$\Leftrightarrow (e^{y^2} \cdot b) - (e^{y^2} \cdot ax) - b = ax$$

$$\Leftrightarrow (e^{y^2} \cdot b) - b = ax + (e^{y^2} \cdot ax)$$

$$\Leftrightarrow (e^{y^2} \cdot b) - b = x\,(a + e^{y^2} \cdot a)$$

$$\Leftrightarrow \quad \frac{(e^{y^2} \cdot b) - b}{a + e^{y^2} \cdot a} = x$$

Df / Umkehrfunktion / Symmetrie Tutorin – Teil 1

$$f(x) = \ln\Big(\underbrace{\tfrac{ax+1}{1-ax}}_{>0}\Big)$$

<u>1 Fall:</u> $\frac{+}{+}$ $\quad ax+1>0 \quad \wedge \quad 1-ax>0$

$\Leftrightarrow \; ax > -1 \qquad \Leftrightarrow \; -ax > -1$

$\boxed{x > -\frac{1}{a}} \qquad \boxed{x < \frac{1}{a}} \qquad x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$

<u>2 Fall:</u> $\frac{-}{-}$ $\quad ax+1<0 \quad \wedge \quad 1-ax<0$

$\Leftrightarrow \; ax < -1 \qquad \Leftrightarrow \; -ax < -1$

$\boxed{x < -\frac{1}{a}} \qquad \boxed{x > \frac{1}{a}} \qquad x \in \left(-\infty, -\frac{1}{a}\right) \cup \left(\frac{1}{a}, \infty\right.$

<u>3 Fall:</u> $1-ax \neq 0$

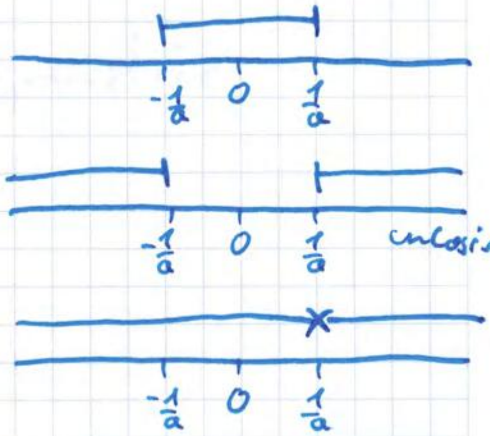
$-ax \neq -1$

$\boxed{x \neq \frac{1}{a}}$

<u>1 Fall:</u>

<u>2 Fall:</u>

<u>3 Fall:</u>



$\Rightarrow x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$

b) Symmetrie

Teil 2

$$f(-x) = \ln\left(\frac{a(-x)+1}{1-a(-x)}\right)$$

$$= \ln\left(\frac{1-ax}{1+ax}\right)$$

$$= \ln\left(\frac{1-ax}{ax+1}\right)$$

$$= \ln\left(\frac{\frac{1}{ax+1}}{1-ax}\right)$$

$$= \ln\left(\frac{ax+1}{1-ax}\right)^{-1}$$

$$= -\ln\left(\frac{ax+1}{1-ax}\right)$$

$f(-x) = -f(x) \Rightarrow$ Punktsymmetrie

c) Umkehrfunktion Teil 3

$$f(x) = \ln\left(\frac{ax+1}{1-ax}\right)$$

$$y = \ln\left(\frac{ax+1}{1-ax}\right)$$

$$\Leftrightarrow e^y = e^{\ln\left(\frac{ax+1}{1-ax}\right)}$$

$$\Leftrightarrow e^y = \frac{ax+1}{1-ax}$$

$$\Leftrightarrow e^y \cdot (1-ax) = ax+1$$

$$\Leftrightarrow e^y - (e^y \cdot ax) = ax+1$$

$$\Leftrightarrow e^y - (e^y \cdot ax) - 1 = ax$$

$$\Leftrightarrow e^y - 1 = ax + e^y \cdot ax$$

$$\Leftrightarrow e^y - 1 = x(a + e^y \cdot a)$$

$$\Leftrightarrow \frac{e^y - 1}{a + e^y \cdot a} = x$$

Df / Umkehrfunktion Tutorium Teil 1

Vollständiges Video:
https://www.youtube.com/watch?v=Yw9Gyv5vyvo

$$f(x) = \ln\left(x+\sqrt{x^2+1}\right)$$

a) Df)

$$\ln\left(x+\sqrt{x^2+1}\right)$$

$$x+\sqrt{x^2+1} > 0 \quad \text{Bedingung}$$

Fall 1: $x \geq 0$

$$\underbrace{x}_{+} + \underbrace{\sqrt{x^2+1}}_{+} \geq 0 \quad \text{o.k.}$$

$$x \in [0, \infty)$$

Fall 2: $x < 0$

$$x + \sqrt{x^2+1} \geq 0$$

$\Leftrightarrow$ ~~[illegible]~~

$\Leftrightarrow \sqrt{x^2+1} \geq -x$, dürfen Quadrieren, weil $-x$ und $x<0$ also, $--x \Rightarrow +x$,

$\Leftrightarrow x^2+1 \geq x^2$

$\Leftrightarrow 1 \geq 0$ o.k. stimmt!

$$x \in (-\infty, 0)$$

$\Rightarrow$ gesamt $Df = \mathbb{R} = (-\infty, \infty)$

b) Umkehrfunktion Teil 2

Berechne f^{-1}

$$y = \ln(x+\sqrt{x^2+1}) \quad | e$$

$$\Leftrightarrow e^y = e^{\ln(x+\sqrt{x^2+1})}$$

$$\Leftrightarrow e^y = (x+\sqrt{x^2+1})$$

$$\Leftrightarrow e^y - x = \sqrt{x^2+1}$$

$$\Leftrightarrow (e^y - x)^2 = x^2 + 1$$

$$\Leftrightarrow (e^y - x)^2 - 1 = x^2$$

$$\Leftrightarrow e^{2y} - 2e^y \cdot x + x^2 - 1 = x^2 \quad | -x^2$$

$$\Leftrightarrow e^{2y} - 2e^y \cdot x - 1 = 0$$

$$\Leftrightarrow e^{2y} - 1 = 2e^y \cdot x \quad | :2e^y$$

$$\Leftrightarrow \frac{e^{2y}-1}{2e^y} = x$$

Symmetrie?
keine Symmetrie
weil der ln auf
~~auf~~ keiner Seite Symmetrisch
ist